Rotational Dynamics and Oscillatory Motion Chapter: 1

Rotational Motion:

The motion of a rigid body which takes place in such a way that all of its particles move in circles about an axis with a common angular velocity. For example;

i. The fixed speed of rotation of the Earth about its axis.
ii. The varying speed of rotation of the flywheel of a sewing machine.
iii. The rotation of a satellite about a planet

Rotation of Rigid Bodies:

We shall analyze the motion of systems of particles and rigid bodies that are undergoing translational and rotational motion. A rigid body is said to be in **translational motion** if each particle of the body has same linear displacement in the equal interval of time. For example; when a bus is moving, then the passengers and the bus itself are in translational motion.

A rigid body is said to be in **rotational motion** about a give axis if each particle of the body has same angular displacement in the equal interval of time. In rotational motion, the different particles of the rigid body have same angular velocity but different linear velocities. For examples, the motion of a wheel of moving bus about its axle is rotational motion.

Moment of Inertia (Rotational Inertia)

The inertia of a body in rotational motion is called **moment of inertia**. The inability of a body to change its state of rest or uniform motion by itself is called **inertia**. A body rotating about an axis has a tendency to be rotating even if a stopping torque is applied to it. Such a property of a rotating body is called rotational inertia or moment of inertia. For example; a rotating fan doesn't stop immediately even if the switch is put off due to rotational inertia.

Let us consider, a body of mass m is rotating about an axis passing through a point O which is at a distance r from the particle as shown in fig. If F be the force applied on the particle, then,

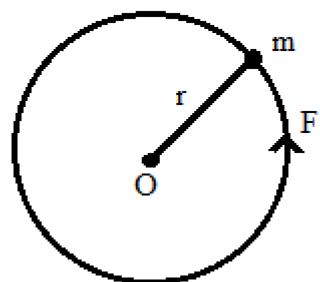


Fig: Moment of inertia

F= ma Where a is the tangential acceleration of the particle.

Also, $a = r\alpha$ Where, α is the angular acceleration of the particle.

Now, torque on the particle due to this force F is

$\tau = r \times F = r \times ma = r \times m(r\alpha)$

$\therefore \tau = (mr^2)\alpha$ ……………..(i)

For a linear motion, we have

Force = linear inertia (mass) × linear acceleration

For a rotational motion, we have

Torque = Rotational inertia (moment of inertia) × angular acceleration

$\therefore \tau = I\alpha$ ……………..(ii)

From eq^n (i) & eq^n (ii), we get

I= mr^2 …………(iii)

Hence, the moment of inertia I of a particle about an axis is measured as the product of its mass and square of its distance from the axis of rotation. In SI unit, I is measured in Kgm^2 & CGS unit, it is expressed in $gmcm^2$. The dimensional formula of I is I= $M \times L^2 = [ML^2T^0]$.

Moment of Inertia Consists of n Particles of a Rigid Body

Let us consider, a rigid body having a mass M is rotating about an axis with constant angular velocity ω. A rigid body consists of n particles of masses m_1, m_2,…..m_n which are at perpendicular distances r_1, r_2,

…..r_n respectively. Then the moment of inertia of these n particles about the axis axis are given by $I_1 = m_1r_1^2,\ I_2 = m_2r_2^2, \dots \dots .. I_n = m_nr_n^2$ respectively.

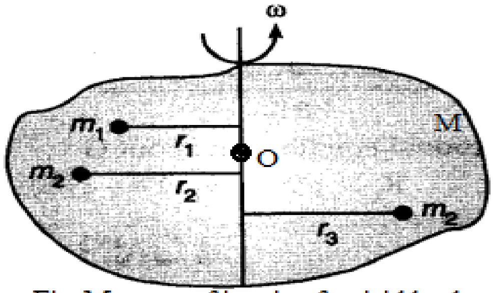


Fig: Moment of inertia of a rigid body

We know that, the moment of inertia I of the rigid body is equal to the sum of moments of inertia of these all the particles.

$I = I_1 + I_2 \dots \dots \dots \dots \dots + I_n$

$I = m_1r_1^2 + m_2r_2^2 \dots \dots \dots .+ m_nr_n^2$

$\therefore I = \sum_{i=1}^{n} m_ir_i^2$

Where, m_i is the mass of i^{th} particle of the body which is at a distance r_i..

Torque or Moment of Force

Torque is a measure of how much a force acting on an object causes that object to rotate. It is denoted by τ. mathematically the torque is define as,

Torque = Force× perpendicular distance of the force from the axis of rotation.

i.e. $\tau = Fr$ ………(i)

where, Torque is a vector quantity, its SI unit is Newton meter (Nm).

It can be expressed as vector form is,

$\vec{\tau} = \vec{r} \times \vec{F}$ …………..(ii)

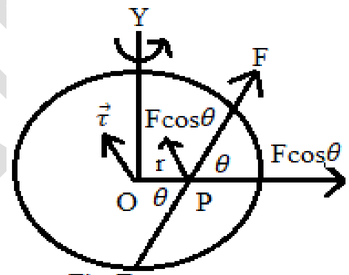


Fig: Torque

The direction of $\vec{\tau}$ is perpendicular to the plane containing $\vec{r}$ and $\vec{F}$. The torque may be clockwise or anti clockwise direction. Its dimensional formula is, $\tau = rF = L \times MLT^{-2} = [ML^2T^{-2}]$

Eq^n (ii) can be written as, $\hat{\tau} = \vec{r} \times \vec{F} = rFsin\theta\hat{n}$ ………..(iii)

Where, θ be angle between $\vec{r}$, $\hat{n}$ and $\vec{F}$.

When $\theta = 0^0$, then eq^n(iii) becomes,

$\tau = Fr\sin 0 = 0$ for minimum

When $\theta = 90^0$, then eq^n(iii) becomes,

$\tau = Fr\sin 90 = Fr$ for maximum

Angular Momentum

The angular momentum of a rigid object is defined as the product of the momentum of inertia and the angular velocity. It is denoted by L. A rotating body possesses angular momentum. It is measured as the product of the linear momentum of a body and the perpendicular distance between the body and the axis of rotation.

Let us consider, a body having a mass m is revolving around a circle of radius r with speed v about an axis passing through the center O. Then, angular momentum of the body is,

L= linear momentum of a body × the perpendicular distance between the body and the axis of rotation.

i.e. L=Pr

or, L=mvr ………(i)   where P=mv

if, ω be the angular velocity of the body, then v = r ω

L=m(r ω)r

L=mr^2 ω ……..(ii)

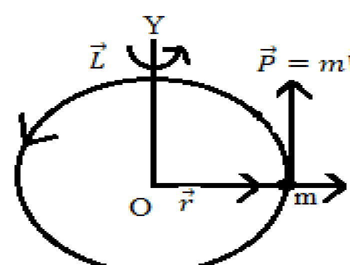


Fig: Body revolving around a circle

Eq^n (i) and eq^n (ii) are the expression for the angular momentum. The SI unit of angular momentum is Kgm^2s^{-1}. The dimensional formula is $[ML^2T^{-1}]$. It is the vector quantity. Which can be expressed as

$\vec{L} = rPsin\theta\hat{n}$ ………………..(iii)

Where θ is the angle between $\vec{r}$ and $\vec{P}$. $\hat{n}$ be unit vector along the direction of $\vec{L}$. The direction of $\vec{L}$ is perpendicular to both $\vec{r}$ and $\vec{P}$.

The magnitude of angular momentum is, L=Psinθ …(iv)

From eq^n(iv) if $\theta = 0^0$

i.e. L=0 for minimum i.e there is no rotational effect on the body

From eq^n(iv) if $\theta = 90^0$

L= rP for maximum i.e. there is maximum rotational effect on the body

Conservation of Angular Momentum:

It states that *'if no external torque acts on a system, then total angular momentum of the system remains conserved.'* i.e. I ω = Constant ……(i)

Where, I is the moment of inertia of a body about a given axis of rotation and ω is its angular velocity.

Proof: Mathematically the torque is define as,

Torque = Force× perpendicular distance of the force from the axis of rotation.

i.e. $\tau = Fr$ ………(ii)

Where, Torque is a vector quantity, its SI unit is Newton meter (Nm).

It can be expressed as vector form is,

$\vec{\tau} = \vec{r} \times \vec{F}$ or, $\vec{\tau} = \vec{r} \times \frac{dP}{dt}$ where, P=mV

$\vec{\tau} = \vec{r} \times \frac{d(m\vec{V})}{dt}$ or, $\vec{\tau} = \frac{d}{dt}(\vec{r} \times m\vec{V})$ where, L= $\vec{r} \times m\vec{V}$

$\vec{\tau} = \frac{d\vec{L}}{dt}$

Therefore, the torque acting on a body is $\vec{\tau} = \frac{d\vec{L}}{dt}$ ………(iii)

If external torque acting is zero. i.e. $\vec{\tau} = 0$

$\frac{d\vec{L}}{dt} = 0$

$d\vec{L} = 0$ Integrating $\int d\vec{L}$= constant

$\vec{L}$ = Constant ………(iv)

From eq^n(i) and eq^n(iv),

I ω =L

This is a principle of conservation of angular momentum.

Rotational Kinetic Energy:

Let us consider, a rigid body having a mass M is rotating about an axis with constant angular velocityω. A rigid body consists of n particles of masses m_1, m_2,…..m_n which are at perpendicular distances r_1, r_2, …..r_n respectively. All of these n particles have same angular velocity ω but different linear velocities.

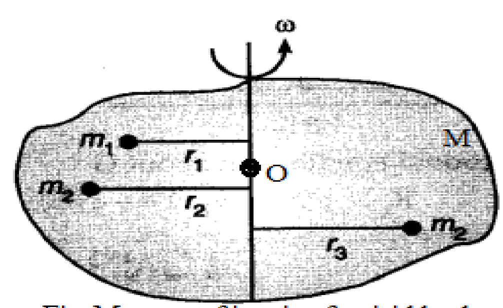


Fig: Moment of inertia of a rigid body

Let. Their respective linear velocities be V_1, V_2, …..V_n. then, $V_1 = r_1\omega$, $V_2 = r_2\omega$……..$V_n = r_n\omega$. The rotational kinetic energy of particle m_1 is,

$E_1 = \frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1(r_1\omega)^2 = \frac{1}{2}m_1r_1^2\omega^2$

Similarly, the rotational K.E. of particles of masses $m_2, m_3 \ldots\ldots\ldots m_n$

$E_2 = \frac{1}{2}m_2r_2^2\omega^2$, $E_3 = \frac{1}{2}m_3r_3^2\omega^2 \ldots\ldots\ldots E_n = \frac{1}{2}m_nr_n^2\omega^2$ respectively.

The rotational K.E. of a rigid body is equal to the sum of K.E. of the particles of the body.

$E_{tot} = E_1 + E_2 + \cdots \ldots \ldots \ldots E_n$

$E_{tot} = \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \frac{1}{2}m_3r_3^2\omega^2 \ldots\ldots\ldots \frac{1}{2}m_nr_n^2\omega^2$

$E_{tot} = \frac{1}{2}(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 \ldots\ldots\ldots m_nr_n^2)\,\omega^2$

$E_{tot} = \frac{1}{2}(\sum mr^2)\,\omega^2$

$E_{tot} = \frac{1}{2}I\,\omega^2$ where, I= mr^2

Simple Harmonic Motion

Periodic Motion: Any motion which repeats itself after regular interval of time is called periodic or harmonic motion. Motion of hands of a clock, motion of earth around the sun, motion of the needle of a sewing machine are the examples of periodic motion.

Oscillatory Motion: If a particles repeats its motion after a regular time interval about a fixed point motion is said to be oscillatory or vibratory, i.e. oscillatory motion is a constrained periodic motion between precisely fixed limits Motion of Piston in an automobile engine, motion of balance wheel of a watch are the examples oscillatory motion.

Time period: Time taken in one complete oscillation is called time period Or, Time after which motion is repeated is called time period.

Frequency: Frequency is the number of oscillations completed by oscillating body in unit time interval. Its SI unit is Hertz.

Simple Harmonic Motion (SHM):

A simple harmonic motion is defined as an oscillatory motion about a fixed point in which the restoring force is always directly proportional to the displacement from that point and is always directed towards that point.

Let the force be F and the displacement of the string from the equilibrium position be x. then,

$F \propto x$

Or, $F = -K\,x$ ………..(i) where k is a proportionality constant.

The negative sign indicated that the restoring force F is develop opposite to the displacement from the mean position.

From Newton's second law of motion, F=ma ……….(ii)

Where, m be mass of particle and a=acceleration

From eq^n (i) and eq^n (ii), $ma = -kx$

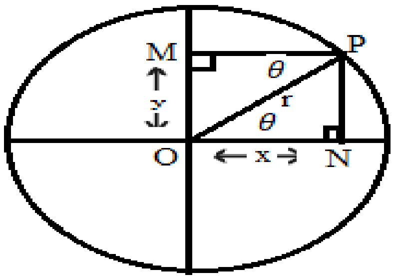
$a = -\frac{k}{m}x$ ……….(iii) Where k and m are constant

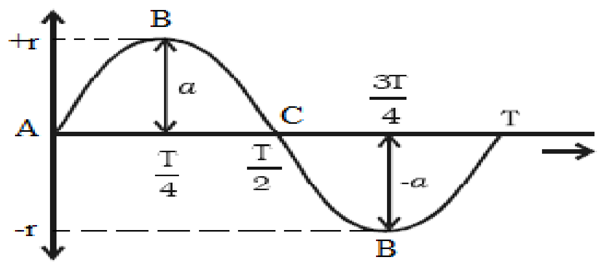
$\therefore a \propto -x$

Above eq^n shows that the acceleration in simple harmonic motion is always directly proportional to the displacement from the mean position and negative sign indicated that it is always directed towards the mean position.

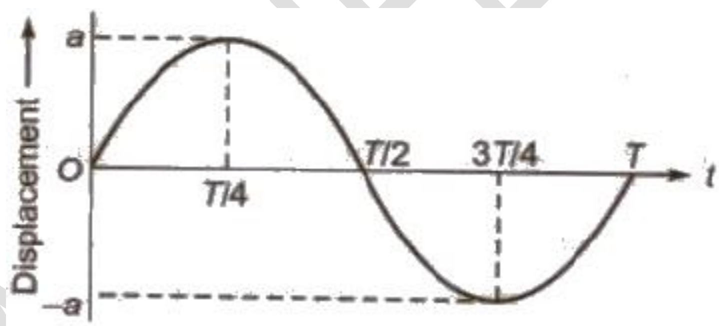
Characteristics of Simple Harmonic Motion (SHM):

The simple harmonic motion has various characteristics. Some of them are described below;





1. **Displacement:** The distance travelled by the vibrating particle at any instant of time t from its mean position is known as displacement. When the particle is at P, the displacement of the particle along Y axis is y as shown in figure. In ΔOPM



$sin\theta = \frac{OM}{OP} = \frac{y}{r}$

$y = rsin\theta$

$\therefore y = rsin\omega t$ where $\theta = \omega t$ ………..(i)

The amplitude of the vibrating particle is defined as its maximum displacement from the mean position. Similarly, In ΔOPN

$cos\theta = \frac{ON}{OP} = \frac{x}{r}$

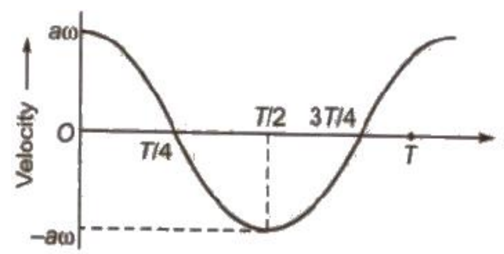
$x = rcos\theta \;\; \therefore x = rcos\;\omega t$ …………(ii)

2. **Amplitude:** from eq^n (i) becomes, $y = rsin\theta$ If y is maximum then $sin\theta = 1$

$\therefore \; y_{max} = r$

Maximum displacement of the particle from the mean position is called amplitude.

3. **Velocity:** The velocity in simple harmonic motion at an instant is defined as the rate of change of displacement at that instant. $V = \frac{dy}{dt} = \frac{d}{dt}(rsin\omega t)$ where $y = rsin\omega t$



$V = r\omega cos\omega t$ …………(iii)

$V = r\omega\sqrt{1 - sin^2\omega t}$

$\therefore \; V = \omega\sqrt{r^2 - y^2}$ ……………..(iv)

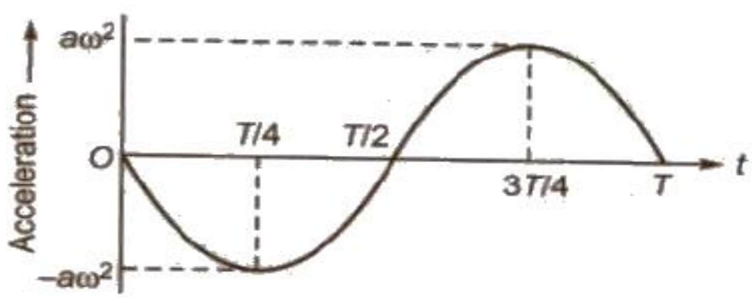
eq^n (iii) and eq^n (iv) are the eq^n for velocity in simple harmonic motion.

At mean position 0; $\therefore y = 0$ and $\therefore V = r\omega$ (maximum value)

At extreme position Y; $\therefore y = r$ and $\therefore V = 0$ (minimum value)

So, a particle in simple harmonic motion has maximum velocity at mean position and minimum velocity at the extreme position.

4. **Acceleration:** The rate of change of velocity is called acceleration.



$a = \frac{dv}{dt} = \frac{d(r\omega cos\omega t)}{dt}$

$a = r\omega\frac{dv}{dt}(cos\omega t) = r\omega(-sin\omega t)\omega$

$a = -\omega^2(rsin\omega t)$

$\therefore a = -\omega^2 y$ …………(v)

This is the eq^n for the acceleration in simple harmonic motion

At mean position 0; $\therefore y = 0$ and $\therefore a = 0$ (minimum value)
At extreme position Y; $\therefore y = r$ and $\therefore a =$ -$r^2\omega$ (maximum value)
So, a particle in simple harmonic motion has zero acceleration at mean position and maximum acceleration at the extreme position.

5. **Time Period (T):** the time taken by the particle to complete one oscillation is called time period in simple harmonic motion. The magnitude of acceleration in simple harmonic motion is given by;

$a = \omega^2 y$ or, $\omega^2 = \frac{a}{y}$ or, $\omega = \sqrt{\frac{a}{y}}$ or, $\frac{2\pi}{T} = \sqrt{\frac{a}{y}}$ or, $T = 2\pi\sqrt{\frac{y}{a}}$ $\therefore T = 2\pi\sqrt{\frac{displacement}{acceleration}}$ ……….(vi)

6. **Frequency;** The number of oscillations made in one second is called frequency in simple harmonic motion. In T seconds, the particle makes 1 oscillation. In 1 seconds, the particle makes $\frac{1}{T}$ oscillation. Frequency = $\frac{1}{Time\ Period}$ $\therefore f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{displacement}{acceleration}}$ …………..(vii)

7. **Wavelength(λ);**

The linear distance travelled by the particle in one oscillation is called wavelength of the particle executing simple harmonic motion. If T be the time period of oscillation and v be the velocity. λ =vT.

8. **Phase;** The phase of the particle executing simple harmonic motion at any instance gives the position and direction of motion of the particle with respect to its mean position. It is measured in terms of fraction of time period T, the fraction angle is 2π. Then, $\therefore y = r sin(\omega t + \emptyset_0)$.

Quantity $(\omega t + \emptyset_0)$ in above equation is known as phase of the motion and the constant $\emptyset_0$ is known as initial phase i.e. phase at time t=0, or phase constant. $\emptyset$ be position from where the time was taken.
Let, $(\omega t + \emptyset_0)$be denoted by $\emptyset$.

$\emptyset = \omega t + \emptyset_0$ or, $\emptyset - \emptyset_0 = \omega t$, So, phase change in time T is, $\emptyset - \emptyset_0 = \omega t = \left(\frac{2\pi}{T}\right)T$

$\therefore \emptyset - \emptyset_0 = 2\pi$

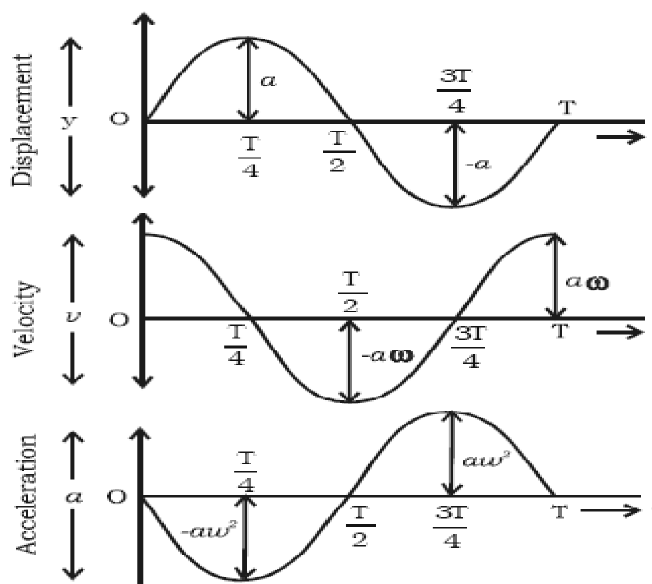
This shows that the phase change in T seconds will be 2π radian which actually means no change in phase.

9. **Graphical representation of displacement, velocity and acceleration in simple harmonic motion;** We have the equations of displacement y, velocity v and acceleration a, in simple harmonic motion are;

$y = r sin\omega t = r sin\left(\frac{2\pi}{T}\right)T, v = r\omega cos\omega t = r\omega cos\left(\frac{2\pi}{T}\right)t$ and

$a = -\omega^2 y = -\omega^2 r sin\omega t = -\omega^2 r sin\left(\frac{2\pi}{T}\right) t$

respectively,
At, t = 0, y = 0, v = rω and a=0
At, t = $\frac{T}{4}$, y = r, v = 0 and a = $-\omega^2 r$
At, t = $\frac{T}{2}$, y = 0, v = -rω and a = 0
At, t = $\frac{3T}{4}$, y = - r, v = 0 and a = $\omega^2 r$
At, t = T, y = 0, v = rω and a = 0
Graphic\l representation of displacement, velocity and acceleration in simple harmonic motion.



Graphical representation

Oscillation of spring:

Let us suppose the spring S with negligible mass which is attached to a wall and the other end to an object of mass m. The spring S with an object are laid on a horizontal table. If the mass is pulled slightly to extend spring and then released, the system vibrates with simple harmonic motion. The center of oscillation O is the position of mass at the end of the string corresponding to it natural length, i.e. when the spring is neither extended nor compressed.

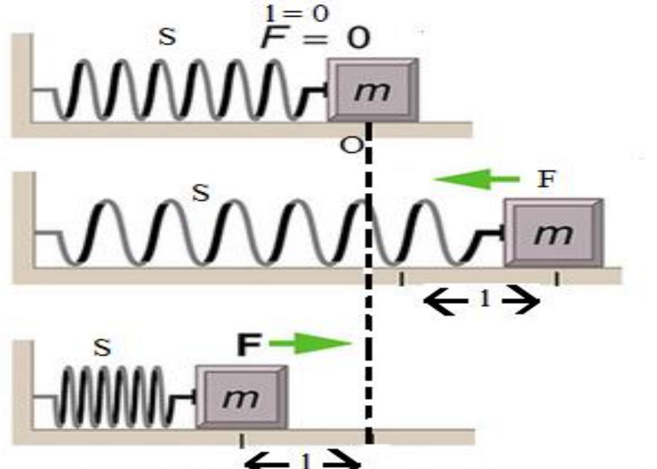


Fig: Horizontal oscillation of mass-spring system

Let L be the extension of the spring and F be the restoring force set up in the spring. Then from Hook's Law. i.e. $F \propto L$

$F = -KL$ ....(i)

Where K is known as spring constant. Negative sign shows that the restoring force acts opposite to the displacement of the mass.

If a is the acceleration produced in the mass, then we have $F = ma$ ....(ii)

Therefore, from eq^n(i) and eq^n(ii), we have

$ma = -KL$ or, $a = -\frac{K}{m} L$............(iii)

$a = -\omega^2 L$......(iv) where $\omega^2 = \frac{K}{m}$ is a constant.

This shows that the acceleration is directly proportional to the displacement and is directed towards the mean position. Hence, the motion of a horizontal mass-spring system is simple harmonic motion.

Expression for time period, T

Comparing eq^n(iii) and eq^n(iv), we get

$\omega^2 = \frac{k}{m}$ Where ω be angular velocity. If T is time period of oscillation, then

$\omega = \sqrt{\frac{K}{m}}$ [i.e. $\omega = 2\pi f = \frac{2\pi}{T}$ $where\ f = \frac{1}{T}$]

$\therefore\ \omega = \frac{2\pi}{T} = \sqrt{\frac{K}{m}}$ where, $\omega = \frac{2\pi}{T}$

$\therefore T = 2\pi\sqrt{\frac{m}{K}}$ ......(v)

Which is the required expression for the time period of oscillation depends upon the mass attached to the spring.

Energy of oscillation Body:

Kinetic Energy (E_K)

Kinetic energy of the particle with velocity v is given by $E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\sqrt{\omega(r^2 - y^2)}\right)^2$ where, v= $\omega(r^2 - y^2)$

$E_k = \frac{1}{2}m(\omega^2 r^2 - y^2)\ ....(i)$

Potential Energy (E_P):

Suppose a particle of mass m is executing simple harmonic motion. (SHM) with amplitude r and angular velocity v. if y is the displacement, then acceleration is, $a = -\omega^2 y$

The restoring force F on the particle is, $F = ma$

$or, F = ma = -m\omega^2 y = -ky,$ Where, $m\omega^2 = k,$ a constant. If a particle is displaced further by a small displacement dy against the force, then work done, $dw = -Fdy = -(-ky)dy = kydy$

Therefore total work done to displace the particle from mean position to the position of displacement y is,

$$w = \int_0^y Fdy = \int_0^y kydy = k\left[\frac{y^2}{2}\right]_0^y = \frac{1}{2}ky^2$$

The work done on the particle will remain in the form of potential energy. Thus,

$E_P = \frac{1}{2}ky^2 \qquad \therefore E_P = \frac{1}{2}\omega^2 y^2$

Total energy of the particle at any points is,

$E = E_P + E_K$ Or, $E = \frac{1}{2}\omega^2 y^2 + \frac{1}{2}\omega^2(r^2 - y^2)$ Or, $E = \frac{1}{2}\omega^2 r^2$

$\therefore E = 2m\pi^2 f^2 r^2$

Here, m,v and r are constant, the total energy remains constant for a particle executing simple harmonic motion.

Special Cases:

Case I: At mean position, y=0

$\therefore E_k = \frac{1}{2}m\omega^2(r^2 - y^2)$

$E_k = \frac{1}{2}m\omega^2(r^2 - 0)$

$E_k = \frac{1}{2}m\omega^2 r^2$

And, $E_P = \frac{1}{2}m\omega^2 y^2$

$E_P = \frac{1}{2}m\omega^2 0^2$

$E_P = 0$

- At mean position, potential energy is zero and kinetic energy is maximum. So, total energy of the particle executing simple harmonic motion at mean position is in the form of kinetic energy.

Case II: At extreme position, y=r:

$\therefore E_k = \frac{1}{2}m\omega^2(r^2 - y^2)$

$E_k = \frac{1}{2}m\omega^2(r^2 - r^2)$

$E_k = \frac{1}{2}m\omega^2 0^2$

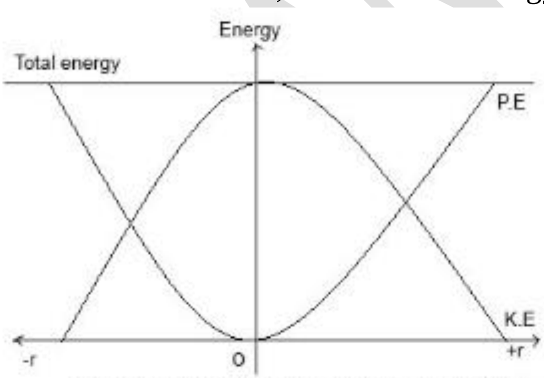
$E_k = 0$

And, $E_P = \frac{1}{2}m\omega^2 y^2$

$E_P = \frac{1}{2}m\omega^2 r^2$

- So, K.E. is zero and P.E. is maximum at extreme position. So, total energy is in the form of P.E. at extreme position of a simple harmonic motion.

The variation of K.E., P.E. and total energy as a function of displacement as shown in fig.



The variation of kinetic energy and potential energy as a function of position in SHM